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On metric spaces with the Haver property which are Menger spaces

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ABSTRACT

A metric space (X,d) has the Haver property if for each sequence $\epsilon_1,\epsilon_2,\ldots$ of positive numbers there exist disjoint open collections $\mathcal{V}_1,\mathcal{V}_2,\ldots$ of open subsets of X, with diameters of members of $\mathcal{V}_i$ less than ϵ_i and $\bigcup_{i=1}^{\infty}\mathcal{V}_i$ covering X, and the Menger property is a classical covering counterpart to σ-compactness. We show that, under Martin's Axiom **MA**, the metric square $(X,d)\times(X,d)$ of a separable metric space with the Haver property can fail this property, even if X^2 is a Menger space, and that there is a separable normed linear Menger space M such that (M,d) has the Haver property for every translation invariant metric d generating the topology of M, but not for every metric generating the topology. These results answer some questions by L. Babinkostova [L. Babinkostova, When does the Haver property imply selective screenability? Topology Appl. 154 (2007) 1971–1979; L. Babinkostova, Selective screenability in topological groups, Topology Appl. 156 (1) (2008) 2–9].

1. Introduction

A metric space (X,d) has the *Haver property* if for each sequence $\epsilon_1,\epsilon_2,\ldots$ of positive numbers there are disjoint open collections $\mathcal{V}_1,\mathcal{V}_2,\ldots$ with diameters of members of $\mathcal{V}_i$ less than ϵ_i and $\bigcup_{i=1}^{\infty}\mathcal{V}_i$ covering X, see [11]. We say that a metrizable space X has the *property C* (or is a *C-space*) if for any metric d generating the topology of X the metric space (X,d) has the Haver property. For some background to these properties we refer the reader to [1,6,2], cf. also Section 2.

There are separable metric spaces with the Haver property which are not C-spaces, cf. Section 2, and there is a separable metric space (X,d) with the Haver property and X being a C-space, such that the metric square $(X,d)\times(X,d)$ fails the Haver property, cf. [17].

However, L. Babinkostova [2] showed that the situation is different in the presence of the Hurewicz property – a classical covering counterpart to σ-compactness.

Let us recall that X has the Menger (respectively, Hurewicz) property if for each sequence $\mathcal{U}_1,\mathcal{U}_2,\ldots$ of open covers of X there are finite open collections $\mathcal{V}_i\subset\mathcal{U}_i$, $i=1,2,\ldots$, such that each $x\in X$ belongs to some (respectively, to all but finitely many) unions $\bigcup\mathcal{V}_i$, cf. [22]. We call such X a Menger (respectively, Hurewicz) space.

Babinkostova proved that if (X,d) has the Haver property and X is a Hurewicz space then X is a C-space, and for any metric space (Y,e) with the Haver property, the metric product $(X,d)\times(Y,e)$ has the Haver property, see [2, Theorem 1 and Corollary 9].

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The following result shows that, at least under Martin's Axiom **MA**, the Hurewicz property cannot be replaced here by the weaker Menger property. This answers some questions asked by Babinkostova in [2, Problems 1 and 2 on page 1977].

Theorem 1.1. *Assuming* **MA**, *there is a separable metric space (X, d) with the Haver property such that X is a C-space and X^2 is a Menger space, but the metric square $(X, d) \times (X, d)$ fails the Haver property.*

Babinkostova [3] considered also some counterparts to the Haver property for topological groups, cf. Section 6, (A), and our second result confirms (under **MA**) a conjecture formulated in [3, Conjecture 3 on page 5].

Theorem 1.2. *Assuming* **MA**, *there is a linear subspace M of the separable Hilbert space which is a Menger space and for each translation invariant metric d generating the topology of M the metric space (M, d) has the Haver property, but M is not a C-space.*

Our constructions combine several ideas from the papers [16,17,20]. An important element in these constructions is the celebrated Michael technique [14] concerning concentrated sets in product spaces. We shall discuss a variation of this technique, suitable for our purposes, in Section 3. Theorem 1.1 is proved in Section 4, and Theorem 1.2 – in Section 5. Some useful preliminaries are explained in the next section and in the last section we gathered several comments on the subject of this paper.

2. Preliminaries

We consider only separable metrizable spaces. Given metric spaces (X, d), (Y, e) we shall denote by $(X, d) \times (Y, e)$ the metric product, i.e. the product $X \times Y$ equipped with the metric $(d \times e)((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), e(y_1, y_2)\}$. Our terminology follows [6]. In particular, a space is countable-dimensional if it is a countable union of zero-dimensional subspaces.

The property C was introduced in Section 1 by a description exhibiting its relations with the metric Haver property. The original, more handy definition, is the following one: a space X has the property C if and only if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X there are disjoint open collections $\mathcal{V}_1, \mathcal{V}_2, \ldots$ with $\mathcal{V}_i$ refining $\mathcal{U}_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ covering X, see [1] and [6] (cf. also [17, Section 4(D)]). We shall call the sequence $\{\mathcal{V}_i\}_{i=1}^{\infty}$ a *C-refinement* of $\{\mathcal{U}_i\}_{i=1}^{\infty}$.

The class of C-spaces is essentially larger than the class of countable-dimensional spaces, cf. [6].

The property C is not hereditary (cf. [7, Example 8.19]), while any subspace of a metric space with the Haver property, endowed with the inherited metric, has the Haver property. In particular, there are metric spaces with the Haver property, which are not C-spaces.

If $\mathcal{U}$ is a collection of subsets of a space X and $Y \subset X$, we write $\mathcal{U} \mid Y = \{U \cap Y\colon U \in \mathcal{U}\}$ and $\bigcup \mathcal{U} = \bigcup\{U\colon U \in \mathcal{U}\}$. Given two collections $\mathcal{U}$, $\mathcal{V}$ of subsets of X we write $\mathcal{U} \prec \mathcal{V}$ if any member of $\mathcal{U}$ is contained in some member of $\mathcal{V}$.

The following observations will be useful in the sequel.

(A) *Let X be a metrizable space and $Y \subset X$. If $\{\mathcal{U}_i\}_{i=1}^{\infty}$ is a sequence of families of open sets in X such that $\bigcup \mathcal{U}_i \supset Y$ and $\{\mathcal{U}_i \mid Y\}_{i=1}^{\infty}$ has a C-refinement in Y, then there is a sequence $\{\mathcal{V}_i\}_{i=1}^{\infty}$ of disjoint open collections in X such that $\mathcal{V}_i \prec \mathcal{U}_i$ and $\bigcup_{i=1}^{\infty} \bigcup \mathcal{V}_i \supset Y$.*

In particular, if Y has the property C, then for every sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of open covers of Y there is a sequence $\{\mathcal{V}_i\}_{i=1}^{\infty}$ of disjoint open collections in X such that $\{\mathcal{V}_i \mid Y\}_{i=1}^{\infty}$ is a C-refinement of $\{\mathcal{U}_i\}_{i=1}^{\infty}$.

This follows from the fact that for every disjoint family $\mathcal{G}$ of sets open in Y there is a disjoint family $\mathcal{V}$ of sets open in X with $\mathcal{V} \mid Y = \mathcal{G}$.

Recall that a set L is a *partition* between disjoint sets A, B in X, if there exist disjoint open sets $U, V \subset X$ such that $X \setminus L = U \cup V$, $A \subset U$ and $B \subset V$. A sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed sets in X is *essential* if for every partitions L_i between A_i and B_i, we have $\bigcap_{i=1}^{\infty} L_i \neq \emptyset$. A space X is called *strongly infinite-dimensional* if it contains an essential sequence of pairs of disjoint closed subsets; otherwise a space is *weakly infinite-dimensional*.

(B) *A sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed sets in X is essential if and only if the sequence $\{\mathcal{U}_i\}_{i=1}^{\infty}$, where $\mathcal{U}_i = \{X \setminus A_i, X \setminus B_i\}$, does not have any C-refinement.*

This fact, noticed in [1], shows that every C-space is weakly infinite-dimensional.

(C) *The product of a space with the Menger property and a σ-compact space has the Menger property, and the product of a C-space by a σ-compact C-space is a C-space.*

For proofs, cf. [10,2,6].

However, neither the property C nor the Menger property is preserved under finite products, cf. [16,19,20,15] and Theorem 6.1 below.

(D) *A closed subspace of a Menger space (respectively, C-space) is a Menger space (respectively, C-space). A countable union of spaces with the Menger property (respectively, with the property C) has the Menger property (respectively, property C).*

(E) *If $N \subset M$ are such that N, as well as each closed subset of M disjoint from N, have the Menger property, then so does M. The same is true for the property C.*

(F) We shall use the following diagonal construction, going back to Hilgers [12], cf. [6, Problem 1.4.F].

Let $p : E \to F$ be a function, let $\mathcal{H}$ be a collection of subsets of E and let $\varphi : B \to \mathcal{H}$ be a surjection defined on some $B \subset F$. The Hilgers function associated with φ is defined as follows: given $b \in B$ we pick $h(b) \in p^{-1}(b) \setminus \varphi(b)$ whenever it is possible and we choose $h(b) \in p^{-1}(b)$ arbitrarily, if $p^{-1}(b) \subset \varphi(b)$. Then, whenever $H \in \mathcal{H}$ contains $\varphi(B)$, we have $p^{-1}(b) \subset H$ for any $b \in \varphi^{-1}(H)$.

We shall also need the following consequences of Martin's Axiom **MA**.

Lemma 2.1. *([8, Proposition 22.I or Corollary 22.J]) If* **MA** *holds, then the intersection of less than continuum G_δ-subsets of the real line (containing the rationals Q) contains a G_δ-set (containing Q).*

Lemma 2.2.

(a) *([9]) Under* **MA***, all sets of cardinality $< 2^\omega$ have the Menger property.*
(b) *([21, Corollary 2.4(2)]) Under* **MA***, a space which is the union of less than 2^ω Menger subspaces, is a Menger space.*

3. Michael's concentrated sets

We shall prove a modification of Lemma 5.2 from [14] to provide a handy tool for our constructions. Our approach is closely related to that of [5,18,4].

Lemma 3.1. *Assume* **MA***. There is a subset B of the irrationals in $I = [0,1]$ intersecting each non-trivial interval in a set of cardinality 2^ω, such that*

(i) *$|B \setminus G| < 2^\omega$ for every dense G_δ-subset G of I,*
(ii) *for any $m > 1$ and each dense G_δ-set G in I^m, $B^m \setminus G$ is contained in the union of the set of points with at least two coordinates equal and a sum of less than 2^ω hyperplanes $x_i = c$.*

Proof. Let $\mathcal{H}_m$ be the family of all dense G_δ-sets in I^m and let us list the elements of $\bigcup_{n=1}^\infty \mathcal{H}_m$ as $\{G_\alpha\colon \alpha < 2^\omega\}$. Thus

(1) for every $\alpha < 2^\omega$ the set G_α is a dense G_δ-set in $I^{m(\alpha)}$ for some $m(\alpha) \in \mathbb{N}$,

and for each $m \in \mathbb{N}$,

(2) for every dense G_δ-set $G \subset I^m$ there is $\alpha < 2^\omega$ with $m(\alpha) = m$ and $G_\alpha = G$.

To shorten the notation we shall write $[m] = \{1,\dots,m\}$.
Given a dense G_δ-set G in I^m and $S \subset [m]$, we set

(3) $G(S) = \{x \in I^S\colon\ G \cap (\{x\} \times I^{[m]\setminus S})$ is dense in $\{x\} \times I^{[m]\setminus S}\}$.

In particular, $G([m]) = G$. For $j \in [m] \setminus S$ and $u \in I^S$ let

(4) $G(u,j) = \{x \in I\colon\ (u,x) \in G(S \cup \{j\})\}$.

By the Kuratowski–Ulam Theorem (see [13, §22, V]), the set $G(S)$ is a dense G_δ-set in I^S and if $u \in G(S)$, then $G(u,j)$ is a dense G_δ-set in I.

Let $\{U_i\colon\ i = 1,2,\dots\}$ be a countable base in I. By the transfinite induction, we will define pairwise disjoint sets $C_\alpha = \{x_i^\alpha\colon\ i = 1,2,\dots\}$, $\alpha < 2^\omega$, so that the following conditions are satisfied for $\alpha < 2^\omega$, $\xi < \alpha$ and $i = 1,2,\dots$:

(5) $x_i^\alpha \in U_i \cap P$, P being the irrationals in I,
(6) $x_i^\alpha \in G_\xi \subset I$, if $m(\xi) = 1$, and $x_i^\alpha \in G_\xi(\{s\}) \subset I$, if $s \in [m(\xi)]$ and $m(\xi) > 1$,
(7) if $m(\xi) > 1$, then for any proper set $S \subset [m(\xi)]$, $j \in [m(\xi)] \setminus S$ and $i \in \mathbb{N}$ we have $x_i^\alpha \in G_\xi(u,j)$ for every $u = (u_{s_1},\dots,u_{s_{|S|}}) \in G_\xi(S)$ with $u_{s_p} \in \bigcup_{\xi<\alpha} C_\xi \cup \{x_k^\alpha\colon\ k < i\}$, $u_{s_p} \neq u_{s_r}$ for $p \neq r$, $p,r \leqslant |S|$.

We begin with an arbitrary countable dense set $C_0 = \{x_i^0\colon\ i = 1,2,\dots\} \subset P$, and at any stage $\alpha > 0$, we shall define points x_i^α inductively on i.

Fix $\alpha < 2^\omega$ and $i \in \mathbb{N}$ and assume that the sets C_ξ, for $\xi < \alpha$, and points x_k^α, for $k < i$, are already defined.

There are less than 2^ω dense G_δ-sets $G_\xi(u,j)$ described in (7) with $\xi < \alpha$, and less than 2^ω sets $G_\xi \subset I$ and sets $G_\xi(\{s\}) \subset I$ described in (6) for $\xi < \alpha$, hence the intersection of all these sets contains a dense G_δ-subset H of I (see Lemma 2.1). Therefore, we can choose $x_i^\alpha \in U_i \cap P \cap H$ such that $x_i^\alpha \notin \bigcup_{\xi<\alpha} C_\xi \cup \{x_k^\alpha\colon\ k < i\}$.

Setting $\Delta(S) = \{u \in I^S\colon$ at least two coordinates of u are equal$\}$, we have

(8) for all $\xi < \alpha < 2^\omega$ and for every $S \subset [m(\xi)]$, $(\bigcup_{\xi<\eta\leqslant\alpha} C_\eta)^S \setminus \Delta(S) \subset G_\xi(S)$.

Indeed, let $u \in (\bigcup_{\xi<\eta\leqslant\alpha} C_\eta)^S \setminus \Delta(S)$, $S = \{s_1, \ldots, s_k\}$. For any $a \in C_\xi$, $b \in C_\eta$, $a \prec b$ means that either (a) $\xi < \eta$ or (b) $\xi = \eta$, $a = x_k^\xi$, $b = x_l^\xi$ and $k < l$. Let $u_1, \ldots, u_k$ be the coordinates of u enumerated so that $u_1 \prec \cdots \prec u_k$. Then $u_1 \in G_\xi(\{s_1\})$, $(u_1, u_2) \in G_\xi(\{s_1, s_2\})$, $\ldots$, $u = (u_1, \ldots, u_k) \in G_\xi(S)$ (see (6)).

We will show now that for $B = \bigcup_{\alpha<2^\omega} C_\alpha$, the conditions (i) and (ii) of our lemma are satisfied. Let G be any dense G_δ-set in some I^m. There exists $\xi < 2^\omega$ such that $m(\xi) = m$ and $G = G_\xi$.

If $m = 1$, we have $\bigcup_{\alpha>\xi} C_\alpha \subset G_\xi$, hence $B \setminus G \subset \bigcup_{\eta\leqslant\xi} C_\eta$, and the last set has cardinality less than 2^ω.

If $m > 1$, consider any $u \in (\bigcup_{\eta>\xi} C_\eta)^{[m]} \setminus \Delta([m])$. Then, by (8), $u \in G([m]) = G$. Thus the set $B^m \setminus G$ consists of points u that either belong to $\Delta([m])$ or have at least one coordinate in the set $\bigcup_{\eta\leqslant\xi} C_\xi$. Therefore the set $B^m \setminus G$ is contained in the union of $\Delta([m])$ and of less than 2^ω hyperplanes $x_k = c$, where $c \in \bigcup_{\eta\leqslant\xi} C_\eta$ and $k \leqslant m$. □

4. Proof of Theorem 1.1

Theorem 1.1 follows readily from the next proposition, where $\vee$ stands for the maximum of two functions, considered on the intersection of their domains.

Proposition 4.1. *Assuming* **MA***, there is a metrizable separable space M which is a union of two C-spaces E_0, E_1, and there exist metrics d_i on E_i generating the topology such that $(E_0 \cap E_1, d_0 \vee d_1)$ fails the Haver property. Moreover, the square $(E_0 \oplus E_1)^2$ of the free union of E_0 and E_1 has the Menger property.*

Indeed, the space $(X, d) = (E_0 \oplus E_1, d)$ – the free union of the metric spaces (E_0, d_0), (E_1, d_1) from Proposition 4.1, has the properties described in Theorem 1.1: the metric space $(E_0 \cap E_1, d_0 \vee d_1)$ (which fails the Haver property) embeds isometrically into the square of (X, d) with the metric $\rho((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), d(y_1, y_2)\}$, so the metric square $(X, d) \times (X, d)$ fails the Haver property and, on the other hand, (X, d) has the Haver property, as each (E_i, d_i) has this property (E_i being a C-space).

Proof of Proposition 4.1. Assume **MA**. Let $K = \prod_{i=0}^\infty I_i$, where $I_i = [0, 1]$, and let $p_n : \prod_{i=0}^\infty I_i \to \prod_{i=0}^n I_i$ be the projection for $n = 0, 1, \ldots$.

Let $C_m = \{(x_0, x_1, \ldots) \in K\colon x_m = 0\}$, $D_m = \{(x_0, x_1, \ldots) \in K\colon x_m = 1\}$, $m = 0, 1, 2, \ldots$, be a sequence of pairs of opposite faces in the Hilbert cube. Recall that this sequence is essential in K and hence, by (B), Section 2, the sequence $\mathcal{U}_m = \{K \setminus C_m, K \setminus D_m\}$ of open covers of K does not have any C-refinement.

To shorten the notation, we set $f = p_0 : K \to [0, 1]$. Then, for every $t \in [0, 1]$,

(1) the fiber $f^{-1}(t) = \{t\} \times \prod_{i=1}^\infty I_i$ cannot be covered by any collection $\bigcup_{j=1}^\infty \mathcal{V}_j$, where $\mathcal{V}_j$ are disjoint families of open sets in K refining $\mathcal{U}_j$, for $j = 1, 2, \ldots$.

Indeed, if there was such a collection, then adding to this collection the family $\mathcal{V}_0 = \{f^{-1}([0, t)), f^{-1}((t, 1])\}$ we would obtain a C-refinement $\mathcal{V}_0, \mathcal{V}_1, \ldots$ of $\mathcal{U}_0, \mathcal{U}_1, \ldots$ in K, which is impossible.

Let B be a set described in Lemma 3.1 ($I = I_0$) and let us split B into pairwise disjoint countable, dense in I sets C_ξ, $\xi < 2^\omega$. Let H_ξ, $\xi < 2^\omega$, be all G_δ-sets in K with $f(H_\xi) \supset B$ and let us consider a Hilgers function $h : B \to K$ for this collection (cf. 2(F)): assuming $c \in C_\xi$, if $f^{-1}(c) \setminus H_\xi \neq \emptyset$, we choose $h(c)$ from this set, and we pick $h(c) \in f^{-1}(c)$ arbitrarily, whenever $f^{-1}(c) \subset H_\xi$.

Setting $E = h(B)$, we have

(2) $\{\mathcal{U}_m \mid E\}_{m=1}^\infty$ does not have any C-refinement in E.

Indeed, the negation of (2) provides disjoint collections $\mathcal{V}_m$, $m = 1, 2, \ldots$, of open sets in K such that $\mathcal{V}_m$ refines $\mathcal{U}_m$ and $\mathcal{V} = \bigcup_{m=1}^\infty \mathcal{V}_m$ covers E (see (A), Section 2). Then $V = \bigcup \mathcal{V}$ is open in K and $f(V) \supset B$, hence $V = H_\xi$ for some $\xi < 2^\omega$. Pick any $t \in C_\xi$. Since $h(t) \in H_\xi$, we have $f^{-1}(t) \subset H_\xi$, which contradicts (1).

We will check that

(3) E has the Menger property.

The proof is similar to that of property (23) in [20]. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots$ be a sequence of open collections in K covering E and let $H = \bigcap_{j=1}^\infty (\bigcup \mathcal{G}_{2j})$. Then H is a G_δ-set in K containing $E = h(B)$. Since $f(E) = B$, we have $f(H) \supset B$, so $H = H_\xi$ for some $\xi < 2^\omega$. For every $c \in C_\xi$, $h(c) \in H_\xi$, hence $f^{-1}(c) \subset H_\xi$ and it follows that $f^{-1}(C_\xi) \subset H_\xi = H$. Let $C_\xi = \{u_1, u_2, \ldots\}$.

For every j, $f^{-1}(u_j) \subset H \subset \bigcup \mathcal{G}_{2j}$, and since $f^{-1}(u_j)$ is compact, there is a finite collection $\mathcal{V}_{2j} \subset \mathcal{G}_{2j}$ such that $f^{-1}(u_j) \subset \bigcup \mathcal{V}_{2j}$. Thus $f^{-1}(C_\xi) \subset \bigcup_{j=1}^{\infty}(\bigcup \mathcal{V}_{2j}) = W$. Since f is closed, one can find an open set $U \subset I$ containing C_ξ such that $f^{-1}(U) \subset W$. Then $E \setminus W = h(B) \setminus W \subset h(B \setminus U)$. By the condition (i) of Lemma 3.1, $|B \setminus U| < 2^\omega$, hence $|E \setminus W| < 2^\omega$ and Lemma 2.2 ensures that $E \setminus W$ has the Menger property. Thus there exist finite collections $\mathcal{V}_{2j-1} \subset \mathcal{G}_{2j-1}$ such that $\bigcup_{j=1}^{\infty} \bigcup \mathcal{V}_{2j-1} \supset E \setminus W$. It follows that $\mathcal{V}_1, \mathcal{V}_2, \ldots$ is a sequence of finite families with $\mathcal{V}_i \subset \mathcal{G}_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ covering E. So E has the Menger property.

Next, as in the construction of Example 2 in [16], let us arrange the set Q of the rationals into a sequence $\{q_1, q_2, \ldots\}$ and let T be the compact space obtained from K by attaching to each fiber $f^{-1}(q_n) = \{q_n\} \times \prod_{i=1}^{\infty} I_i$ the cube $\{q_n\} \times \prod_{i=1}^{n} I_i$ by the mapping $p_n \mid f^{-1}(q_n)$.

Let

(4) $\pi : K \to T$, $T_n = \pi(\{q_n\} \times \prod_{i=1}^{\infty} I_i)$

be the quotient map and the attached Euclidean cubes, respectively.

Since $f : K \to I_0$ is constant on every fiber of the mapping $\pi : K \to T$ it induces a continuous mapping $\tilde{f} : T \to I_0$ with $f = \tilde{f} \circ \pi$. We set $\tilde{E} = \pi(E)$ and $e_b = \pi(h(b))$ for $b \in B$ (notice that $\tilde{f}(e_b) = b$).

Next we define, finally, the sets $E_0, E_1 \subset T$. Let us split $\mathbb{N}$ into two disjoint infinite sets N_0 and N_1 such that both sets $Q_s = \{q_i\colon\ i \in N_s\}$, $s = 0, 1$, are dense in Q.

We set

(5) $S_s = \bigcup\{T_j\colon\ j \in N_s\}$, $E_s = \tilde{E} \cup S_s$, for $s = 0, 1$, and $M = E_0 \cup E_1$.

We will show that

(6) E_s has the property C, for $s = 0, 1$.

Indeed, let $\mathcal{G}_0, \mathcal{G}_1, \ldots$ be a sequence of open covers of $E_s = \tilde{E} \cup S_s$. First, using the fact that S_s is countable-dimensional, we find a sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of families of disjoint open subsets of T such that $\mathcal{W}_i \mid E_s$ refines $\mathcal{G}_i$ for $i = 1, 2, \ldots$ and $W = \bigcup_{i=1}^{\infty} \bigcup \mathcal{W}_i \supset S_s$ (see (A), Section 2). Since $\tilde{f} : T \to I_0$ is perfect, there is a set U open in I_0 such that $U \supset Q_s$ and $\tilde{f}^{-1}(U) \subset W$. By (i) of Lemma 3.1, $|B \setminus U| < 2^\omega$, therefore the set $E_s \setminus W$ has cardinality $< 2^\omega$, hence it is at most 0-dimensional. Thus we can find a family $\mathcal{W}_0$ of disjoint open subsets of T covering $E_s \setminus W$ and such that $\mathcal{W}_0 \mid E_s$ refines $\mathcal{G}_0$. It follows that the family $\{\mathcal{W}_i \mid E_s\}_{i=0}^{\infty}$ is a C-refinement of $\{\mathcal{G}_i\}_{i=0}^{\infty}$.

Since $\pi \mid E \to \tilde{E}$ is a homeomorphism and S_s is σ-compact, we infer from (3) and (D), Section 2, that

(7) $\tilde{E}$ and E_s have the Menger property, for $s = 0, 1$.

We shall show that there are metrics d_s on E_s generating the topology such that

(8) $(\tilde{E}, d_0 \vee d_1)$ fails the Haver property,

where $d_0 \vee d_1$ is the maximum of the metrics d_s restricted to $\tilde{E} = E_0 \cap E_1$, cf. (5).

The idea of the proof is taken from [17]. For each m, fix open sets $V_0(m), V_1(m)$ in T such that, for $s = 0, 1$,

(9) if $F_s(m) = \bigcup\{T_j\colon\ j \in N_s,\ j \leqslant m\}$ then $F_s(m) \subset V_s(m)$, $\overline{V_0(m)} \cap \overline{V_1(m)} = \emptyset$.

Let us consider

(10) $A_m = \pi(C_m)$, $B_m = \pi(D_m)$.

Since $p_n(C_m) \cap p_n(D_m) = \emptyset$ for $n \geqslant m$, we have by (4),

(11) $A_m \cap B_m \subset \bigcup_{j \leqslant m} T_j = F_0(m) \cup F_1(m)$.

It follows, cf. (9) and (11), that $A_m \cap V_{1-s}(m) \cap E_s$ and $B_m \cap V_{1-s}(m) \cap E_s$ have disjoint closures in E_s. Therefore, there are continuous maps $\varphi_m^s : E_s \to [0, 1]$ such that

(12) $\varphi_m^s \mid A_m \cap V_{1-s}(m) \equiv 0$, $\varphi_m^s \mid B_m \cap V_{1-s}(m) \equiv 1$.

Let ρ be any metric on T generating the topology. We set, for $s = 0, 1$,

(13) $d_s(x, y) = \rho(x, y) + \sum_{m=1}^{\infty} 2^{-m} |\varphi_m^s(x) - \varphi_m^s(y)|$,

where $x, y \in E_s$. Then d_s is a metric generating the topology of the space E_s.

Since $\pi \mid E: E \to \tilde{E}$ is a homeomorphism, by (2),

(14) the sequence $\tilde{\mathcal{U}}_m = \{\tilde{E} \setminus A_m, \tilde{E} \setminus B_m\}$, $m = 1, 2, \ldots$, has no C-refinement in $\tilde{E}$.

Therefore, to show that $(\tilde{E}, d_0 \vee d_1)$ fails the Haver property it is enough to check that for each m there is $\epsilon_m > 0$ such that

(15) $(d_0 \vee d_1)(a, b) \geqslant \epsilon_m$, whenever $a \in A_m \cap \tilde{E}$, $b \in B_m \cap \tilde{E}$.

Indeed, (14) and (15) imply that there are no disjoint open collections $\mathcal{V}_m$ in $\tilde{E}$ with $d_0 \vee d_1$-diameters of sets in $\mathcal{V}_m$ less than ϵ_m and $\bigcup_m \mathcal{V}_m$ covering $\tilde{E}$, because this would provide a C-refinement of $\{\tilde{\mathcal{U}}_m\}_{m=1}^{\infty}$, contrary to (14).

We now define the ϵ_m's as required in (15). Let us fix m, and let us consider open in T sets $W_s(m)$ such that, cf. (9),

(16) $F_s(m) \subset W_s(m) \subset \overline{W_s(m)} \subset V_s(m)$.

Let $\delta_s > 0$ be the ρ-distance between the sets $\overline{W_s(m)}$ and $T \setminus V_s(m)$, and let $\delta > 0$ be the ρ-distance between $A_m \setminus (W_0(m) \cup W_1(m))$ and $B_m \setminus (W_0(m) \cup W_1(m))$ in T. We define

$$\epsilon_m = \min\{\delta_0, \delta_1, \delta, 2^{-m}\}.$$

To verify that these ϵ_m's are as required, let us take any $a \in A_m \cap \tilde{E}$, $b \in B_m \cap \tilde{E}$. We consider several cases.

If both a, b are outside of $W_0(m) \cup W_1(m)$, then $(d_0 \vee d_1)(a, b) \geqslant \rho(a, b) \geqslant \delta$, cf. (13).

Suppose that $a \in W_0(m)$. If $b \notin V_0(m)$, then $(d_0 \vee d_1)(a, b) \geqslant \rho(a, b) \geqslant \delta_0$. If $b \in V_0(m)$, we have $a \in A_m \cap V_0(m)$ and $b \in B_m \cap V_0(m)$. Then, by (12), $\varphi_m^1(a) = 0$, $\varphi_m^1(b) = 1$, and hence $(d_0 \vee d_1)(a, b) \geqslant d_1(a, b) \geqslant 2^{-m}$, cf. (13).

Suppose that $a \in W_1(m)$. If $b \notin V_1(m)$, then $(d_0 \vee d_1)(a, b) \geqslant \rho(a, b) \geqslant \delta_1$. If $b \in V_1(m)$, we have $a \in A_m \cap V_1(m)$ and $b \in B_m \cap V_1(m)$. Then, by (12), $\varphi_m^0(a) = 0$, $\varphi_m^0(b) = 1$, and hence $(d_0 \vee d_1)(a, b) \geqslant d_0(a, b) \geqslant 2^{-m}$, cf. (13).

If $b \in W_0(m) \cup W_1(m)$, then we proceed similarly.

Namely, if $b \in W_0(m)$, then $(d_0 \vee d_1)(a, b) \geqslant \delta_0$ for $a \notin V_0(m)$, and $(d_0 \vee d_1)(a, b) \geqslant 2^{-m}$ for $a \in V_0(m)$.

Finally, if $b \in W_1(m)$, then $(d_0 \vee d_1)(a, b) \geqslant \delta_1$ for $a \notin V_1(m)$, and $(d_0 \vee d_1)(a, b) \geqslant 2^{-m}$ for $a \in V_1(m)$.

This justifies that our ϵ_m satisfies (15), and completes the proof of (8).

Finally, we prove that $(E_0 \oplus E_1)^2$ has the Menger property. We have to show that for every $s, t \in \{0, 1\}$,

(17) $E_s \times E_t$ has the Menger property.

The space $S_s \times S_t$ is σ-compact, so, by (E) in Section 2, to prove (17) it suffices to show that

(18) every set F which is closed in $E_s \times E_t$ and disjoint from $S_s \times S_t$ has the Menger property.

Let V be an open subset of $T \times T$ such that $V \cap (E_s \times E_t) = (E_s \times E_t) \setminus F$. Let $g = \tilde{f} \times \tilde{f}: T \times T \to I_0 \times I_0$. Since $S_s \times S_t = g^{-1}(Q_s \times Q_t)$ and g is perfect, there is a set G open in $I_0 \times I_0$ containing $Q_s \times Q_t$ such that $g^{-1}(G) \subset V$. Since B satisfies the condition (ii) from Lemma 3.1, the set $B^2 \setminus G$ is contained in a set $H_1 \cup H_2 \cup \Delta$, where $\Delta = \{(x_1, x_2) \in I_0^2\colon x_1 = x_2\}$ and $H_i = \{(x_1, x_2) \in I_0^2\colon x_i \in A_i\}$, with $A_i \subset B$ and $|A_i| < 2^{\omega}$, for $i = 1, 2$.

Thus F is contained in the union of the family $\mathcal{H}$ consisting of the set $Y = ((E_s \times E_t) \setminus V) \cap g^{-1}(\Delta)$ and the sets of the form $(E_s \times E_t) \cap g^{-1}(H)$, where $H = \{(x_1, x_2) \in I^2\colon x_i = b\}$ for $b \in A_i \cup Q$ and $i = 1, 2$. Since, by Lemma 2.2(b), the union of less than 2^{ω} Menger spaces is a Menger space, by (D) from Section 2 it suffices to show that every member of $\mathcal{H}$ is a Menger space.

To that end, first notice that the set $Y \subset ((E_s \times E_t) \setminus g^{-1}(Q_s \times Q_t)) \cap g^{-1}(\Delta)$ is a closed subset of the space $\{(e_b, e_b)\colon b \in B\}$ homeomorphic to the space $\tilde{E}$. Since $\tilde{E}$ is a Menger space by (7), so is Y (see (D) in Section 2).

Now, let $H = \{(x_1, x_2) \in I^2\colon x_i = b\}$, where $b \in A_i \cup Q$ and $i = 1, 2$. If $b \in B$, then $(E_s \times E_t) \cap g^{-1}(H)$ is homeomorphic to $\{e_b\} \times E_t$ (if $i = 1$) or to $E_s \times \{e_b\}$ (if $i = 2$). For $b = q_j \in Q$, $(E_s \times E_t) \cap g^{-1}(H)$ is homeomorphic to $T_j \times E_t$ if $i = 1$ or to $E_s \times T_j$ if $i = 2$. Since E_s and E_t are Menger spaces, by (7), and T_j is compact, we see that in all these cases H is a Menger space (see (C) in Section 2).

This ends the proof of (18) and completes the proof of Proposition 4.1. □

5. Proof of Theorem 1.2

We shall derive Theorem 1.2 from the following proposition. The proof of this proposition is a refinement of some reasonings already used in the proof of Proposition 4.1, cf. also [20]. However, for the reader's convenience we decided to give full explanations, repeating, if needed, some arguments.

Proposition 5.1. *Assuming* **MA**, *there is a separable metrizable space T and its subspace S such that*

(i) T^m *has the property C for* $m = 1, 2, \dots$,
(ii) S^m *has the Menger property for* $m = 1, 2, \dots$,
(iii) S *fails the property C.*

Proof. Let B be a set with the properties (i) and (ii) in Lemma 3.1. Let us split B into 2^ω pairwise disjoint, countable dense sets in I and let $\mathcal{C}$ denote the collection of these sets. Furthermore, let us decompose $\mathcal{C}$ into pairwise disjoint families $\mathcal{C}_1, \mathcal{C}_2, \dots$ with $|\mathcal{C}_m| = 2^\omega$, and let us split each $C \in \mathcal{C}_m$ into pairwise disjoint m-element sets $F(C, i)$, $i = 1, 2, \dots$ such that

(1) $E_m(C) = \bigcup\{F(C, i)^m\colon\ i = 1, 2, \dots\}$ is dense in I^m.

We set

(2) $E_m = \bigcup\{E_m(C)\colon\ C \in \mathcal{C}_m\}$,
(3) $E^\star_m = \{(x_1, \dots, x_m) \in E_m\colon\ x_1 < \dots < x_m\}$.

Let us consider the Hilbert cube

(4) $K = \prod_{i=0}^\infty I_i$, $I_i = [0, 1]$,

and the maps

(5) $p_0 : K \to I_0$, $p_0(x_0, x_1, \dots) = x_0$ and $P_m = p_0 \times \dots \times p_0 : K^m \to I^m_0$.

For each m, let $\mathcal{H}_m = \{H(u)\colon\ u \in E^\star_m\}$ be the collection of all G_δ-sets H in K^m with $P_m(H) \supset E^\star_m$, where $H(u) = H(v)$ for u and v in the same set $E_m(C)$, cf. (1). Let us consider a Hilgers function $h_m : E^\star_m \to K^m$ for the collection $\mathcal{H}_m$ (cf. 2(F)), i.e., we choose $h_m(u) \in P^{-1}_m(u) \setminus H(u)$, whenever such choice is possible, and an arbitrary element $h_m(u) \in P^{-1}_m(u)$, if $P^{-1}_m(u) \subset H(u)$ (cf. the definition of E in the proof of Proposition 4.1).

Let us notice that

(6) for every G_δ-set H in K^m, if $h_m(E^\star_m) \subset H$, then there exists a countable set A dense in $E^\star_m$ such that $P^{-1}_m(A) \subset H$.

Indeed, if $h_m(E^\star_m) \subset H$, then $E^\star_m \subset P_m(H)$, and we have $H = H(u)$ for some $u \in E^\star_m$. Let $C \in \mathcal{C}_m$ be such that $u \in E_m(C)$. Then $A = E_m(C) \cap E^\star_m$ satisfies (6), as for every $v \in A$, $h_m(v) \in H = H(v)$ and hence $P^{-1}_m(v) \subset H$.

For any $C \in \mathcal{C}_m$ and $u = (x_1, \dots, x_m) \in F(C, i)^m$ with $x_1 < \dots < x_m$, we write

(7) $h_m(x_1, \dots, x_m) = (s(x_1), \dots, s(x_m))$,

where $s(x_i) \in p^{-1}_0(x_i)$, cf. (5).

The formula (7) defines a function $s : \bigcup\mathcal{C}_m \to K$, as for any $x \in \bigcup\mathcal{C}_m$ there is a unique $C \in \mathcal{C}_m$ containing x and hence a unique point $u \in E^\star_m$ such that x is a coordinate of u. Since the unions $\bigcup\mathcal{C}_m$ are pairwise disjoint and cover B, we get in effect a function $s : B \to K$ determined by (7). Let

(8) $S = \{s(x)\colon\ x \in B\}$.

We shall verify that

(9) S^m is a Menger space, for $m = 1, 2, \dots$.

To see this for $m = 1$, let us consider a sequence $\mathcal{G}_1, \mathcal{G}_2, \dots$ of open collections in K such that each $\mathcal{G}_i$ covers S. Then $H = \bigcap_{j=1}^\infty(\bigcup\mathcal{G}_{2j})$ is a G_δ-set in K containing $h_1(E^\star_1)$, and by (6), there exists a countable dense subset A of $E^\star_1 = E_1$ such that $P^{-1}_1(A) \subset H$. Since E_1 is dense in I_0, so is $A = \{a_1, a_2, \dots\}$. For every j, the set $P^{-1}_1(a_j) = p^{-1}_0(a_j)$ is compact. Let $\mathcal{W}_{2j} \subset \mathcal{G}_{2j}$ be a finite collection such that $P^{-1}_1(a_j) \subset \bigcup\mathcal{W}_{2j}$. Then $P^{-1}_1(A) \subset \bigcup_{j=1}^\infty(\bigcup\mathcal{W}_{2j}) = W$. Since P_1 is perfect, there is an open set $U \subset I_0$ containing A such that $f^{-1}(U) \subset W$. From the condition (i) in Lemma 3.1, $|B \setminus U| < 2^\omega$, hence $S \setminus W = h_1(B) \setminus W \subset h_1(B \setminus U)$ has cardinality less than 2^ω and hence it is a Menger space, cf. Section 2(D). It follows that there exist finite collections $\mathcal{W}_{2j-1} \subset \mathcal{G}_{2j-1}$ such that $\bigcup_{j=1}^\infty \bigcup\mathcal{W}_{2j-1} \supset S \setminus W$. In effect, we get a sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of finite families $\mathcal{W}_i \subset \mathcal{G}_i$ such that $\bigcup_{i=1}^\infty \mathcal{W}_i$ covers S.

Suppose now that (9) is true for $m - 1$. Observe that $s(x) \in p^{-1}_0(x)$ for each $x \in B$, and hence every point $(s(x_1), \dots, s(x_m))$ from S^m belongs to $P^{-1}_m(x_1, \dots, x_m)$.

For $1 \leqslant i < j \leqslant m$ let

(10) $K^m_{ij} = \{(x_1, \ldots, x_m) \in I^m_0\colon\ x_i = x_j\}$

and let

(11) $\Delta(m) = \bigcup\{K^m_{ij}\colon\ 1 \leqslant i < j \leqslant m\}$.

First, let us note that

(12) $Y_1 = P_m^{-1}(\Delta(m)) \cap S^m$ is a Menger space.

Indeed, for every $i < j \leqslant m$, the space $S^m \cap P_m^{-1}(K^m_{ij})$ is homeomorphic to S^{m-1}, hence it is a Menger space by the inductive assumption. Thus Y_1 is a finite union of Menger subspaces and so it has the Menger property, which completes the proof of (12).

Let $J = \{(x_1, \ldots, x_m) \in I^m_0\colon\ x_1 < \cdots < x_m\}$. Since $S^m \setminus Y_1$ is obtained from $Y_2 = S^m \cap P_m^{-1}(J)$ by permutations of the coordinates, $S^m \setminus Y_1$ is a union of finitely many subspaces homeomorphic to Y_2. Therefore, by (12) and Section 2(D), to prove that S^m is a Menger space it suffices to show that

(13) Y_2 is a Menger space.

To prove (13) we proceed as in (9). We consider an arbitrary sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$ of collections of open in K^m coverings of Y_2, and we infer from $h_m(E^\star_m) \subset Y_2 \subset H = \bigcap_{j=1}^\infty (\bigcup \mathcal{G}_{2j})$ that there is a countable dense set A in $E^\star_m$ with $P_m^{-1}(A) \subset H$. Next, since P_m is perfect, we find finite subcollections $\mathcal{W}_{2i} \subset \mathcal{G}_{2i}$ and an open set $G \subset I^m_0$ containing A, with $P_m^{-1}(G) \subset W = \bigcup_{i=1}^\infty \bigcup \mathcal{W}_{2i}$.

Since $E^\star_m = E_m \cap J$, by the condition (ii) of Lemma 3.1, the set $(B^m \setminus G) \cap J$ is contained in the union of less than 2^ω hyperplanes $L = \{(x_1, \ldots, x_m) \in I^m_0\colon\ x_j = c\}$. Since $P_m^{-1}(L) \cap S^m$ is homeomorphic to $p_0^{-1}(c) \times S^{m-1}$, it is a Menger space, by the inductive assumption. Thus $Y_2 \setminus W$ is a closed subset of a union of less than 2^ω Menger spaces $P_m^{-1}(L) \cap S^m$, and it is a Menger space, by Lemma 2.2. Therefore, one can find finite families $\mathcal{W}_{2i-1} \subset \mathcal{G}_{2i-1}$ such that $Y_2 \setminus W \subset \bigcup_{i=1}^\infty \bigcup \mathcal{W}_{2i-1}$. In effect, $\mathcal{W}_1, \mathcal{W}_2, \ldots$ is a sequence of finite families such that $\mathcal{W}_i \subset \mathcal{G}_i$ and $\bigcup_{i=1}^\infty \mathcal{W}_i$ covers Y_2. This demonstrates (13) and ends the proof of (9).

We will now show that

(14) S fails the property C.

To that end, we set $C_m = \{(x_0, x_1, \ldots) \in K\colon\ x_m = 0\}$, $D_m = \{(x_0, x_1, \ldots) \in K\colon\ x_m = 1\}$ for $m = 0, 1, 2, \ldots,$ and let $\mathcal{U}_j = \{K \setminus C_j, K \setminus D_j\}$ for $j = 1, 2, \ldots.$ Then, for every fiber $p_0^{-1}(t) = \{t\} \times \prod_{i=1}^\infty I_i$,

(15) $\{\mathcal{U}_j \mid p_0^{-1}(t)\}_{j=1}^\infty$ has no C-refinement,

cf. Section 2 and condition (1) in Section 4.

Striving for a contradiction, assume that S has the property C. Then there exist disjoint open families $\mathcal{V}_j \prec \mathcal{U}_j$ such that $V = \bigcup_{j=1}^\infty \bigcup \mathcal{V}_j \supset S$. We have $h_1(E_1) \subset V$, as $B \subset p_0(V)$, and by (6), there is a point t in E_1 with $p_0^{-1}(t) \subset V$. Then $p_0^{-1}(t)$ is covered by $\bigcup_{j=1}^\infty \mathcal{V}_j$, hence $\{\mathcal{V}_j \mid p_0^{-1}(t)\}_{j=1}^\infty$ is a C-refinement of $\{\mathcal{U}_j \mid p_0^{-1}(t)\}_{j=1}^\infty$, contrary to (15). This demonstrates (14).

We shall now extend S to a space T in the following way. Let $q_1, q_2, \ldots$ be the rationals in I_0, and let us attach to each compactum $K_n = \{q_n\} \times \prod_{i=1}^\infty I_i$, the Euclidean cube $\{q_n\} \times \prod_{i=1}^n I_i$ by the projection $p_n(q_n, x_1, x_2, \ldots) = (q_n, x_1, x_2, \ldots, x_n)$. Let $\pi : K \to K^\star$ be the quotient map onto the space $K^\star$ resulting from this operation, and let $T_n = \pi(K_n)$. We shall identify S with its homeomorphic image $\pi(S)$ in $K^\star$, and we set

(16) $T = S \cup Z \subset K^\star$, where $Z = \bigcup_{i=1}^\infty T_i$.

Let $\tilde{p}_0 : K^\star \to I_0$ be the map induced by the projection, i.e., $p_0 = \tilde{p}_0 \circ \pi$. We will show by induction that, for $m = 1, 2, \ldots,$

(17) if A is a space of cardinality less than 2^ω, then $A \times T^m$ is a C-space.

Let $m = 1$. Since Z is a countable union of Euclidean cubes and $\dim A = 0$, the product $A \times Z$ is countable-dimensional and hence it has the property C. Thus, by (E) in Section 2, it suffices to check that for every set W which is open in $A \times K^\star$ and contains $A \times Z$, the space $(A \times T) \setminus W$ has the property C. For every $a \in A$, the set W contains $\{a\} \times Z = \{a\} \times (\tilde{p}_0)^{-1}(Q)$, and since $\tilde{p}_0$ is perfect, there is an open $G_a \subset I_0$ containing Q such that $\{a\} \times (\tilde{p}_0)^{-1}(G_a) \subset W$. Since $|A| < 2^\omega$, by **MA** there exists a G_δ-set G in I_0 containing Q such that $G \subset \bigcap_{a\in A} G_a$ (see Lemma 2.1). Since B satisfies the condition (i) of Lemma 3.1, $|B \setminus G| < 2^\omega$, and since $A \times (\tilde{p}_0)^{-1}(G) \subset W$, $|(A \times T) \setminus W| < 2^\omega$. In particular, the set $(A \times T) \setminus W$ is 0-dimensional, and so it has the property C.

Suppose now that $m > 1$ and (17) is true for all $n < m$. The product $A \times Z^m$ is countable-dimensional (and hence it is a C-space), as A is 0-dimensional and Z^m is countable-dimensional. Therefore, by (E), Section 2, to get (17) it is enough to show that

(18) if $A \times Z^m \subset W$ and W is open in $A \times (K^\star)^m$, then $(A \times T^m) \setminus W$ is a C-space.

For every $a \in A$ the set $W_a = (\{a\} \times (K^\star)^m) \cap W$ is an open subset of $\{a\} \times (K^\star)^m$ containing $\{a\} \times Z^m = \{a\} \times (\tilde{p}_0^m)^{-1}(Q^m)$, hence there is a set G_a open in I_0^m such that $G_a \supset Q^m$ and $\{a\} \times (\tilde{p}_0^m)^{-1}(G_a) \subset W_a \subset W$. By **MA**,

(19) there exists a G_δ-set $G \subset I_0^m$ containing Q^m such that $G \subset \bigcap_{a\in A} G_a$.

Indeed, there exists a perfect surjection $g : P \to I_0^m \setminus Q^m$ defined on the space of irrationals P. For every $a \in A$, the set $I_0^m \setminus G_a$ is σ-compact, hence the set $F_a = g^{-1}(I_0^m \setminus G_a)$ is a σ-compact subset of P. Since $|A| < 2^\omega$, by **MA** (see [8, Proposition 22.I or Corollary 22.J], cf. Lemma 2.1) there exists a σ-compact subset F of P containing $\bigcup_{a\in A} F_a$. Thus $G = I^m \setminus g(F)$ is a G_δ-subset of I^m containing Q^m and contained in $\bigcap_{a\in A} G_a$.

From (19) and the definition of G_a it follows that

(20) $(A \times T^m) \cap W \supset A \times (\tilde{p}_0^m)^{-1}(G) \supset A \times Z^m$.

By the condition (ii) in Lemma 3.1, $B^m \setminus G$ is contained in the union $\Delta(m) \cup \bigcup_{i=1}^m L_i$, where $\Delta(m)$ consists of points of I_0^m with at least two coordinates equal and $L_i = \{(x_1, \ldots, x_m) \in I_0^m\colon\ x_i \in B_i\}$, $B_i \subset B$ being a set of cardinality less than 2^ω, $i \leqslant m$.

Therefore, by (20), the set $(A \times T^m) \setminus W$ is a closed subset of the union $X \cup \bigcup_{i=1}^m Y_i \cup \bigcup_{i=1}^m Z_i$, where

(21) $X = A \times (T^m \cap (\tilde{p}_0^m)^{-1}(\Delta(m)))$,
(22) $Y_i = A \times (T^m \cap (\tilde{p}_0^m)^{-1}(L_i))$,
(23) $Z_i = A \times (T^m \cap (\tilde{p}_0^m)^{-1}(M_i))$ and $M_i = \{(x_1, \ldots, x_m) \in I_0^m\colon\ x_i \in Q\}$.

Thus, to prove (18), it suffices to show that X, Y_i and Z_i are C-spaces, for $i \leqslant m$.

First note that $T^m \cap (\tilde{p}_0^m)^{-1}(\Delta(m)) \subset \bigcup\{K_{ij}\colon\ i < j \leqslant m\} \cup \bigcup\{M_{ij}\colon\ i < j \leqslant m\}$, where $K_{ij} = \{(z_1, \ldots, z_m) \in T^m\colon\ z_i = z_j\}$ and $M_{ij} = \{(z_1, \ldots, z_m) \in T^m\colon\ \tilde{p}_0(z_i), \tilde{p}_0(z_j) \in Q\}$.

Since K_{ij} is homeomorphic to T^{m-1}, by the inductive assumption, $A \times K_{ij}$ is a C-space. The space M_{ij} is homeomorphic to $(\tilde{p}_0)^{-1}(Q) \times (\tilde{p}_0)^{-1}(Q) \times T^{m-2}$, and $(\tilde{p}_0)^{-1}(Q) = Z$ is a countable union of Euclidean cubes (see (16)). Therefore, the inductive assumption yields that $A \times T^{m-2}$ is a C-space, hence by (C) in Section 2, $A \times M_{ij}$ is a C-space. Thus X is a C-space as a closed subset of a union of finitely many C-spaces.

The set $T^m \cap (\tilde{p}_0^m)^{-1}(L_i)$ is homeomorphic to $(\tilde{p}_0)^{-1}(B_i) \times T^{m-1}$, and since $|A \times (\tilde{p}_0)^{-1}(B_i)| < 2^\omega$, the inductive assumption guarantees also that Y_i has the property C.

Finally, the set $T^m \cap (\tilde{p}_0^m)^{-1}(M_i)$ is homeomorphic to $T^{m-1} \times Z$, cf. (16). In effect Z_i is homeomorphic to $A \times T^{m-1} \times Z$ and hence it is a C-space by the inductive assumption and (C) in Section 2. $\square$

Proof of Theorem 1.2. We proceed as in [18, Section 2.6]. Let S and T be the spaces described in Proposition 5.1, and let $\sigma : K^\star \to l_2$ embeds $K^\star$ onto a linearly independent subspace of the separable Hilbert space l_2. We can assume that S is dense in T.

Let L, M be the linear spans of $\sigma(T)$ and $\sigma(S)$, respectively.

The assertion of Proposition 5.1 and a reasoning in [18], Section 2.6, yield the following:

(24) L has the property C,
(25) M has the Menger property,
(26) $\sigma(S)$ is a closed subspace in M and M is dense in L.

By (26), M fails the property C.

Let d be any translation invariant metric on M generating the topology. We shall see that (M, d) has the Haver property.

To that end, let us consider a sequence $\epsilon_1, \epsilon_2, \ldots$ of positive numbers, let U_n be the open ball in the metric space (M, d) centered at zero with radius $\frac{\epsilon_n}{3}$, and let $U_n = M \cap V_n$, where V_n is a neighbourhood of zero in L. We set

(27) $\mathcal{U}_n = \{x + U_n\colon\ x \in M\}$, $\mathcal{W}_n = \{x + V_n\colon\ x \in M\}$.

Taking into account that

(28) $x + (A \cap M) = (x + A) \cap M$, for any $A \subset L$ and $x \in M$,

we have $\mathcal{W}_n \mid M = \mathcal{U}_n$, and since $\bigcup \mathcal{W}_n = M + V_n$, and M is dense in L, cf. (26), $M + V_n = L$, i.e., $\mathcal{W}_n$ covers L. The property C of L provides disjoint open collections $\mathcal{V}_n$ in L with $\mathcal{V}_n$ refining $\mathcal{W}_n$ and $\bigcup_n \mathcal{V}_n$ covering L. Then $\mathcal{V}_n \mid M$ refines $\mathcal{U}_n$ and $\bigcup_n (\mathcal{V}_n \mid M)$ covers M. Finally, d being translation invariant, the diameters of elements of $\mathcal{V}_n$ are less than ϵ_n. □

6. Comments

(A) Babinkostova [3] defined the following property $S_c(\mathcal{O}_{nbd}, \mathcal{O})$ of topological groups G: for any sequence $U_1, U_2, \ldots$ of neighbourhoods of zero in G there is a sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots$ such that $\mathcal{B}_n$ is a pairwise disjoint open family refining $\mathcal{U}_n = \{x \cdot U_n\colon\ x \in G\}$ and $\bigcup_n \mathcal{B}_n$ covers G. For metrizable G, $S_c(\mathcal{O}_{nbd}, \mathcal{O})$ is equivalent to the property that (G, d) has the Haver property for any left invariant metric generating the topology of G, cf. [3, Theorem 2].

Let us notice also that this last property is equivalent to the condition that there exists a left invariant metric e on G such that (G, e) has the Haver property, cf. [3, Section 5]. Indeed, for any left invariant metrics e, d on G generating the topology, the identity map is uniformly continuous as a map from (G, e) to (G, d) and hence the Haver property of (G, e) yields the Haver property of (G, d).

(B) As proved in [16], under the assumption of the Continuum Hypothesis **CH**, there exists a metrizable separable C-space X such that the product $X \times P$ of X by the space of irrationals P is strongly infinite-dimensional, hence it is not a C-space. The construction given in Section 4 provides the following theorem.

Theorem 6.1. *Assuming* **MA**, *there exists a C-space X, whose square X^2 has the Menger property, such that the product $X \times P$ of X with the space of irrationals P is strongly infinite-dimensional, and hence it is not a C-space.*

Indeed, any space $X = E_s$, $s = 0, 1$, constructed in the proof of Theorem 4.1, has such properties. The space E_s is a C-space by (13) and $E_s \times E_s$ has the Menger property by (24).

Consider the mapping $\hat{f} = \tilde{f} \mid E_s : E_s \to I_0$, cf. Section 4. Then $\hat{f}^{-1}(P) = \tilde{E}$ and $E_s \times P$ contains as a closed subspace the graph $\{(x, \hat{f}(x))\colon\ x \in \tilde{E}\}$ of $\hat{f} \mid \tilde{E}$. Now, the graph is homeomorphic to $\tilde{E}$ and hence it is homeomorphic to a strongly infinite-dimensional space E, cf. (2) in Section 4 and (B) in Section 2. In effect, $E_s \times P$ is strongly infinite-dimensional.

One can also show that the square $X \times X = E_s \times E_s$ of the space X is a C-space.

(C) Let us note that the space M constructed in Theorem 1.2, equipped with a translation-invariant metric d, as well as the space $E = E_0 \cap E_1$ defined in Theorem 4.1, equipped with the metric inherited from E_0, are examples of metric separable spaces with the Haver property which are Menger spaces without the property C.

This shows that under **MA**, Problem 4 on page 1979 in [2] has a positive answer.

Moreover, applying some reasonings used in Section 4, one can prove the following.

Proposition 6.2. *Assuming* **MA**, *for each compact metrizable space X which fails the property C there is $E \subset X$ with the Menger property and without the property C, such that (E, d) has the Haver property for some metric d on E generating the topology.*

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